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# ENVELOPES OF ONE-PARAMETER FAMILIES OF PLANE CURVES

BY W. J. RISLEY AND W. E. MACDONALD

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If, in the region  $|x - x_0| < h$ ,  $|a - a_0| < k$ , 1)  $f(x, a)$  is analytic; 2)  $f_a(x, a) \neq 0$ ; 3)  $f_a(x_0, a_0) = 0$ ; 4)  $f_{a,x}(x_0, a_0) \neq 0$ ; 5)  $f_{a,a}(x_0, a_0) \neq 0$ ; then the family of curves,  $y = f(x, a)$ , has an envelope  $E$ , determined by the equations a)  $f_a(x, a) = 0$ , b)  $y - f(x, a) = 0$ .

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$$\text{but} \quad f_{a,x^m}(x_0, a) \neq 0,$$

$$2) \quad f^{a^j}(x, a_0) \equiv 0, \quad j = 1, 2, 3, \dots, n,$$

$$\text{but} \quad f_{a^{n+1}}(x, a_0) \neq 0;$$

then if

$$3) \quad f_{a^{n+1},x^m}(x_0, a_0) \neq 0,$$

the curves  $y = f(x, a)$  have no envelope in the neighborhood of  $(x_0, y_0)$ , except a point envelope which occurs wherever  $m \geq 1$ ; but if

$$4) \quad f_{a^{n+1}, x^m}(x_0, a_0) = 0,$$

the family  $y = f(x, a)$  has an envelope composed of one or more curves through the point  $(x_0, y_0)$ ; also, whenever  $m \geq 1$ , a point envelope at that point.

When  $m = 0$ , the notation  $i = 0, 1, 2, \dots, m - 1$  is meaningless and 1) must be replaced by  $f_a(x_0, a) \neq 0$ .

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$$1) \quad F(x_0, y_0, a_0) = 0, \quad 2) \quad F_a(x_0, y_0, a_0) = 0,$$

$$3) \quad \frac{D(F, F_a)}{D(x, y)} \equiv \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F_a}{\partial x} & \frac{\partial F_a}{\partial y} \end{vmatrix} \neq 0 \text{ at } (x_0, y_0, a_0),$$

$$4) \quad F_{a,a}(x_0, y_0, a_0) \neq 0;$$

then the curves  $F(x, y, a) = 0$  have an envelope  $x = \phi(a), y = \psi(a)$ , which consists, in the neighborhood of  $(x_0, y_0, a_0)$ , of a single analytic curve through this point.

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$$1) \quad F = 0, \quad 2) \quad F_x = 0, \quad 3) \quad F_y = 0, \quad 4) \quad F_{xx} F_{yy} - F_{xy}^2 \neq 0.$$

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$$4a) \quad F_{xx} F_{yy} - F_{xy}^2 = 0, \quad 4b) \quad F_{xx} F_{yy} - F_{xy}^2 > 0,$$

respectively.

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$$1) \quad F = 0, \quad 2) \quad F_x = 0, \quad 3) \quad F_y = 0, \quad 4) \quad F_{xx} F_{yy} - F_{xy}^2 = 0,$$

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$$1) \quad F \equiv 0, \quad 2) \quad F_x \equiv 0, \quad 3) \quad F_y \equiv 0, \quad 4) \quad F_a \equiv 0,$$

$$5) \quad F_{ax} \neq 0 \text{ or } F_{ay} \neq 0, \quad 6) \quad F_{xx} F_{yy} - F_{xy}^2 \neq 0,$$

$$7) \quad \begin{vmatrix} F_{xx} & F_{xy} & F_{xa} \\ F_{yx} & F_{yy} & F_{ya} \\ F_{ax} & F_{ay} & F_{aa} \end{vmatrix} \equiv 0$$

be satisfied along this locus.

*Theorem II.* p. 95. Sufficient conditions that the given locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a nodal locus of the curves  $F(x, y, a) = 0$ , are that the relations

$$1) F \equiv 0, \quad 2) F_x \equiv 0, \quad 3) F_y \equiv 0, \quad 4) F_{xx} F_{yy} - F_{xy}^2 \neq 0$$

be satisfied along this locus.

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If, furthermore, 6)  $F(\mu, \nu, a) \equiv 0$ , 7)  $F_{ax}(\mu, \nu, a) \neq 0$ , or  $F_{ay}(\mu, \nu, a) \neq 0$ , then  $|\mu'(a)| + |\nu'(a)| \neq 0$ , and  $x = \mu(a)$ ,  $y = \nu(a)$  define, in the neighborhood of  $(x_0, y_0, a_0)$ , a nodal locus of the curves  $F(x, y, a) = 0$ .

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*Theorem I.* p. 98. Necessary conditions that a given locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a locus of double points with coincident tangents for the curves  $F(x, y, a) = 0$ , are that 1)  $F \equiv 0$ , 2)  $F_x \equiv 0$ , 3)  $F_y \equiv 0$ , 4)  $F_a \equiv 0$ , 5)  $F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya} \equiv F_{ax} : F_{ay} : F_{aa}$ , along this locus, while 6)  $F_{xx}, F_{xy}, F_{yy}$  do not all vanish simultaneously at any point of the given locus.

*Theorem II.* p. 99. Sufficient conditions that a given locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a locus of double points with coincident tangents for the curves  $F(x, y, a) = 0$ , are that 1)  $F \equiv 0$ , 2)  $F_x \equiv 0$ , 3)  $F_y \equiv 0$ , 4)  $F_{xx} F_{yy} - F_{xy}^2 \equiv 0$ , along the locus, while 5)  $F_{xx}, F_{xy}, F_{yy}$  do not all vanish simultaneously at any point of the given locus.

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14. Examples. p. 100.

THE purpose of this paper is to discuss the conditions under which a family of plane curves, dependent upon a single parameter, will possess an envelope. The discussion is exhaustive for the case that the equation of the curves of the family is given in the explicit form  $y = f(x, a)$ ,  $f(x, a)$  being analytic in  $x, a$ . It may be noted at the outset that the whole treatment is applicable whether the variables are real or complex, unless otherwise specifically stated. The first part of the paper deals with the equation in the solved form :

$$y = f(x, a),$$

and the second part with the equation in the unsolved form :

$$F(x, y, a) = 0.$$

## PART I. THE EXPLICIT CASE.

**1. Definition of an Envelope.** Let a one-parameter family of plane curves be defined by the equation :

$$(1) \quad y = f(x, a);$$

where  $f(x, a)$  is an analytic function of the two independent variables  $x$  and  $a$  in the neighborhood of a point  $(x_0, a_0)$ , i. e., for

$$|x - x_0| < h, \quad |a - a_0| < k;$$

and where

$$f'_a(x, a) \neq 0$$

in this neighborhood.

If there exists a curve  $E$ , defined by the equations :

$$(2) \quad x = \phi(a), \quad y = \psi(a),$$

where  $\phi(a)$  and  $\psi(a)$  are analytic functions of  $a$  in the region  $|a - a_0| < k' \leq k$ , and where  $|\phi'(a)| + |\psi'(a)| > 0^*$  for  $a$  in the same region, and if, for an arbitrary value  $a'$  of  $a$  in the neighborhood of  $a_0$ , the curve  $y = f(x, a')$  is tangent to the curve  $E$  at the point  $a'$ , i. e., at the point  $x' = \phi(a')$ ,  $y' = \psi(a')$ , then the curve  $E$  is said to be an envelope of the curves of the family.

**2. Necessary Conditions.** For any  $a = a'$  in the neighborhood considered, the point  $a'$  of the envelope must be a point of the curve

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\* This requirement excludes the so-called point-envelopes.

$y = f(x, a')$ , and furthermore the slopes of these two curves at this point must be the same; these facts lead to the two identities in  $a$ :

$$(3) \quad \psi(a) - f[\phi(a), a] \equiv 0,$$

$$(4) \quad \psi'(a) - \phi'(a)f_x[\phi(a), a] \equiv 0.$$

From differentiation of (3) with respect to  $a$  there results the identity:

$$(5) \quad \psi'(a) - \phi'(a)f_x[\phi(a), a] - f_a[\phi(a), a] \equiv 0,$$

which, subtracted from (4), gives

$$(6) \quad f_a[\phi(a), a] \equiv 0.$$

The inference is, then, that if an envelope as defined above exists, the two functions  $x = \phi(a)$  and  $y = \psi(a)$  must satisfy the equations:

$$(7) \quad y - f(x, a) = 0,$$

$$(8) \quad f_a(x, a) = 0,$$

which constitute, therefore, necessary conditions.

**3. Sufficient Conditions.** A set of sufficient conditions may be stated and proved as follows:

**THEOREM.** *If*

1)  $f(x, a)$  is an analytic function of  $x$  and  $a$  in the region

$$|x - x_0| < h, \quad |a - a_0| < k;$$

$$2a) \quad f_a(x, a) \neq 0,$$

in the same region; but

$$2b) \quad f_a(x_0, a_0) = 0;$$

$$3) \quad f_{a,x}(x_0, a_0) \neq 0;$$

$$4) \quad f_{a,a}(x_0, a_0) \neq 0;$$

then the one parameter family of curves defined by  $y = f(x, a)$  has an envelope  $E$  which is determined by the two equations:

$$a) \quad f_a(x, a) = 0,$$

$$b) \quad y - f(x, a) = 0.$$

*Proof.* From the continuity of the functions, conditions 3) and 4) show that  $f_{a,x}(x, a) \neq 0$  and  $f_{a,a}(x, a) \neq$  throughout a certain neighborhood of  $(x_0, a_0)$  such as  $|x - x_0| < h' \leq h$ ,  $|a - a_0| < k' \leq k$ . Then by virtue of conditions 2b) and 3), equation a) may be solved for  $x$ , by the ordinary Implicit Function Theorem,\* giving  $x = \omega(a)$ , where  $\omega(a)$  is analytic in the region  $|a - a_0| < k'' \leq k'$  and reduces to  $x_0$  for  $a = a_0$ . Substitution of  $\omega(a)$  for  $x$  in b) yields  $y = \chi(a)$ , where  $\chi(a)$  is analytic in the same region and reduces to  $y_0$  for  $a = a_0$ .

These functions,

$$x = \omega(a), \quad y = \chi(a),$$

satisfy the two equations a) and b); thus

$$(9) \quad f_a[\omega(a), a] \equiv 0, \quad \chi(a) - f[\omega(a), a] \equiv 0,$$

two identities in  $a$  for  $|a - a_0| < k''$ . Differentiating the first of these we find that

$$(10) \quad f_{a,x}[\omega(a), a] \omega'(a) + f_{a,a}[\omega(a), a] \equiv 0,$$

from which, together with 3) and 4), it is clear that

$$(11) \quad \omega'(a) \equiv -\frac{f_{a,a}(x, a)}{f_{a,x}(x, a)} \neq 0, \neq \infty, \quad x = \omega(a).$$

The slope of the curve  $x = \omega(a)$ ,  $y = \chi(a)$  at any point  $a$  is given by the relation:

$$(12) \quad \frac{dy}{dx} = \frac{\chi'(a)}{\omega'(a)}.$$

The slope of the curve of the family  $y = f(x, a)$ , for this value of  $a$  and at any of its points, is determined by the equation:

$$(13) \quad \frac{dy}{dx} = f_x(x, a).$$

But the second identity of (9) yields on differentiation:

$$(14) \quad \chi'(a) - f_x[\omega(a), a] \omega'(a) - f_a[\omega(a), a] \equiv 0,$$

which, with the first of identities (9), shows that

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\* Osgood: *Lehrbuch der Funktionentheorie*, p. 48, vol. 1, and p. 345. The proof given for algebraic equations holds for the more general type here considered.

$$(15) \quad \frac{\chi'(a)}{\omega'(a)} \equiv f_x(x, a), \quad x = \omega(a),$$

thus proving the slopes identical at the arbitrary point  $x = \omega(a)$ ,  $y = \chi(a)$ .

Since, by (11),  $\omega'(a) \neq 0$ , it follows that the condition  $|\omega'(a)| + |\chi'(a)| > 0$  is satisfied and the curve above determined meets all the requirements of our definition of an envelope.

**4. Some Limitations of these Conditions.** The conditions of the foregoing article are sufficient to give an envelope which, at and near a given point  $a_0$ , is composed of a single analytic branch. They are not necessary even for this restricted case. Let us consider two examples.

*Example 1.* Does the family of curves

$$y = (x - a)^2$$

have an envelope which is smooth in and about the point  $a = a_0 = 0$ ,  $x = x_0 = 0$ ?

Here

$$\begin{aligned} f_a(x, a) &\equiv -2(x - a) \neq 0, & f_a(0, 0) &= 0; \\ f_{a,x}(0, 0) &= -2 \neq 0, & f_{a,a}(0, 0) &= 2 \neq 0. \end{aligned}$$

These being satisfied, it is assured that the curves in question have an envelope smooth at the point  $x_0 = 0$ ,  $a_0 = 0$ , and that this envelope is obtained by solving the equations:

$$a) \quad f_a(x, a) \equiv -2(x - a) = 0,$$

$$b) \quad y - f(x, a) \equiv y - (x - a)^2 = 0,$$

for  $x$  and  $y$  in terms of  $a$ . The solution is immediate and the envelope of the system is

$$y = 0, \quad (x = a).$$

*Example 2.* Does the family of curves

$$y = (x - a)^3$$

have an envelope which is smooth in and about the point  $a = a_0 = 0$ ,  $x = x_0 = 0$ ? Conditions 1) and 2) are fulfilled just as before but 3) and 4) are violated, for

$$f_{a,x}(x_0, a_0) = -6(x_0 - a_0) = 0, \quad f_{a,a}(x_0, a_0) = 6(x_0 - a_0) = 0.$$

Geometrically it is evident that this family of curves also has an envelope which is quite as smooth as that of the curves of Example 1, being, indeed, identical with it. In fact, the family

$$y = (x - a)^n, \quad n = 3, 4, \dots$$

has  $y = 0$  as an envelope though conditions 3) and 4) both fail.

Since some of the hypotheses of the theorem fail in cases where envelopes are known to exist, it is clear that the theorem contains hypotheses that are unnecessarily restrictive.

**5. Weierstrass's Theorem on Implicit Functions.** The narrowness of the theorem of §3 arises from the weakness of the Implicit Function Theorem employed for the solution of the equation  $f_a(x, a) = 0$ . This theorem gives conditions sufficient, but so restrictive as to leave untouched a number of cases which it is essential to be able to treat. As a means of handling these cases use is made of the Implicit Function Theorem of Weierstrass, which may be stated as follows:

**WEIERSTRASS'S THEOREM.** *Let  $F(x, y)$  be a function of the two independent variables  $x$  and  $y$ , analytic in the neighborhood of a point  $(x_0, y_0)$ ; let*

$$F(x_0, y) \neq 0,$$

and let

$$F(x_0, y_0) = 0;$$

then the function  $F(x, y)$  may be written thus:

$$F(x, y) \equiv P(x, y) \quad H(x, y),$$

where

$$P(x, y) \equiv (y - y_0)^n + A_1(x)(y - y_0)^{n-1} + \dots + A_n(x),$$

the  $A_i(x)$  being analytic functions of  $x$  in the neighborhood of  $x_0$  and vanishing for  $x = x_0$ , and  $H(x, y)$  being an analytic function of  $x$  and  $y$  in the neighborhood of  $(x_0, y_0)$ , not vanishing at  $(x_0, y_0)$ .\*

The inference is, therefore, that the equation  $F(x, y) = 0$  has  $n$ , and only  $n$ , roots  $y$  as functions of  $x$  which approach  $y_0$  as  $x$  approaches  $x_0$ , and that these roots are the  $n$  roots of  $P(x, y) = 0$ . Attention may thus be

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\* For the proof of this theorem, the reader is referred to Goursat: *Cours d'analyse*, vol. 2, pp. 280-285. See also Picard, *Traité d'analyse*, vol. 2, p. 243; Bliss, *Bull. Amer. Math. Soc.*, vol. 16, p. 356 (April, 1910); MacMillan, *ibid.*, vol. 17, p. 116 (Dec. 1910).

restricted in a study of the behavior of the roots of  $F(x, y) = 0$  in the neighborhood in question to a consideration of the roots of the simpler equation  $P(x, y) = 0$ .

To the discussion of this equation the following considerations are essential.

*Definition.* The function

$$P(x, y) \equiv y^n + A_1(x)y^{n-1} + A_2(x)y^{n-2} + \cdots + A_n(x),$$

where  $A_i(x)$  is analytic at  $x = x_0$  and  $A_i(x_0) = 0$ , is *reducible* at the point  $(x = x_0, y = y_0 = 0)$  if

$$P(x, y) \equiv [y^m + B_1(x)y^{m-1} + \cdots + B_m(x)][y^l + C_1(x)y^{l-1} + \cdots + C_l(x)],$$

where  $m > 0$ ,  $l > 0$ , and  $B_j(x)$  and  $C_k(x)$  satisfy the same conditions as  $A_i(x)$ . Otherwise it is *irreducible*.

**THEOREM.** *A necessary and sufficient condition that  $P(x, y)$  be irreducible at the point  $(x_0, y_0 = 0)$  is that, no matter how small a neighborhood of the point  $x_0$  one consider, the Riemann surface for the function  $y$  of  $x$  defined by  $P(x, y) = 0$  spread out over this neighborhood consist of a single piece.*

*Proof.* 1. *The condition is sufficient.* For if  $P(x, y)$  be reducible the Riemann surface for  $y$  as a function of  $x$  defined by  $P(x, y) = 0$  will, in the neighborhood of  $x_0$ , be composed of the separate Riemann surfaces corresponding to the separate irreducible factors.\*

2. *The condition is necessary.* The equation  $P(x, y) = 0$  defines  $y$  as an  $n$ -valued function of  $x$  in the neighborhood of  $x = x_0$ ; let the branches be denoted by  $f_1(x), f_2(x), \dots, f_n(x)$ . Assume that the Riemann surface breaks up and let one of the pieces be a  $\lambda$ -sheeted piece,  $\lambda < n$ , to which are assigned the branches  $f_1(x), \dots, f_\lambda(x)$ . Sever the surface by a cut running out indefinitely from the point  $x = x_0$ , and let the description of a closed circuit around  $x = x_0$  by the independent variable cause  $f_1(x)$  to go over into  $f_2(x)$ ,  $f_2(x)$  into  $f_3(x)$ , and so on,  $f_\lambda(x)$  going over into  $f_1(x)$ . Consider then the function

$$\begin{aligned}\bar{P}(x, y) &\equiv [y - f_1(x)][y - f_2(x)] \cdots [y - f_\lambda(x)] \\ &\equiv y^\lambda + \beta_1(x)y^{\lambda-1} + \cdots + \beta_\lambda(x).\end{aligned}$$

The functions  $\beta_i(x)$  are integral symmetric functions of the  $f$ 's, and con-

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\* See, in this connection, Osgood: *Lehrbuch der Funktionentheorie*, vol. 1, p. 357.

sequently are functions of  $x$ , single-valued and analytic throughout the unsevered neighborhood of  $x = x_0$ , except possibly at  $x_0$  itself. Moreover  $\beta_i(x)$  is finite and continuous in this region, and  $\beta_i(x_0) = 0$ . Hence  $\beta_i(x)$  is analytic throughout the unsevered neighborhood of  $x = x_0$  inclusive of the point itself.\* For if a function is analytic throughout the neighborhood of a point, with the possible exception of the point itself, and if it remains finite in the neighborhood, then it approaches a limit as the variable approaches the point; furthermore if this limit is assigned as the value of the function at the point, then the function is analytic at this point also. These conditions are fulfilled in the present case. The function  $\bar{P}(x, y)$  is therefore irreducible at the point  $x = x_0$  (by part 1 of this proof).

Hence it is seen that  $P(x, y)$  admits as a factor an irreducible polynomial,  $\bar{P}(x, y)$ , of the same type. This contradicts the hypothesis that  $P(x, y)$  is irreducible, and the theorem is proved.

Equating to zero each of the irreducible factors of  $P(x, y)$  we have  $y$  defined as a function of  $x$ , single or multiple valued as the case may be. In case the function corresponding to a given irreducible factor is multiple valued, the branches are all connected in a single cycle at  $x_0$ . In this case all the values of  $y$  corresponding to an arbitrary  $x \neq x_0$  in a suitably restricted neighborhood of  $x_0$  are distinct; for the discriminant of the irreducible factor is a function of  $x$ , analytic in the neighborhood of the point  $x_0$  and vanishing there, but not vanishing identically. It has, therefore, no second root in the vicinity of  $x_0$ . Furthermore unless this function  $y$  of  $x$  is a constant, the inverse function exists and  $x$  can be expressed as a single valued function of  $y$  in the neighborhood of  $y_0$ , or as a multiple valued function whose branches are connected in cycle at  $y_0$ . There will, of course, be as many such functions  $x$  as there are irreducible factors in  $P(x, y)$ .

## 6. Application of the Theorem of the Preceding Article.

A necessary condition for an envelope of the curves  $y = f(x, a)$  has been seen to consist in the equation

$$f_a(x, a) = 0.$$

We now proceed to inquire how far this condition is sufficient.

The function  $f_a(x, a)$  may contain certain easily detected factors which are extraneous for our present purposes, factors, namely, of either or both of the forms

$$(x - x_0)^m, \quad (a - a_0)^n,$$

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\* See Osgood: *Funktionentheorie*, vol. 1, p. 262, Riemannscher Satz.

where  $m$  and  $n$  are positive integers. If  $f_{a^{n+1}}(x, a_0) \neq 0$ , but all the preceding  $a$ -derivatives of  $f_a(x, a)$ —if there are any—vanish identically for  $a = a_0$ , then  $f_a(x, a)$  contains the factor  $(a - a_0)^n$  but not the factor  $(a - a_0)^{n+1}$ . If  $f_{a,x^m}(x_0, a) \neq 0$ , but all the preceding  $x$ -derivatives of  $f_a(x, a)$ —if there are any—vanish identically for  $x = x_0$ , then  $f_a(x, a)$  contains the factor  $(x - x_0)^m$  but not the factor  $(x - x_0)^{m+1}$ . Hence we may write

$$f_a(x, a) \equiv (a - a_0)^n (x - x_0)^m F(x, a),$$

where  $m, n$  are determined as just indicated, and where

$$F(x, a_0) \neq 0, \quad F(x_0, a) \neq 0.$$

The solutions of the equation

$$f_a(x, a) = 0$$

are the solutions of the three separate equations,

$$\begin{aligned} a) \quad & a - a_0 = 0, \\ b) \quad & x - x_0 = 0, \\ c) \quad & F(x, a) = 0. \end{aligned}$$

Equation  $a)$  does not define  $x$  as a function of  $a$ ; hence the factor  $(a - a_0)^n$  gives rise to no envelope.

Equation  $b)$  gives  $x = x_0$ . Writing  $f_a(x, a)$  as

$$\begin{aligned} f_a(x, a) &\equiv (x - x_0)^m (a - a_0)^n F(x, a) \\ &\equiv (x - x_0)^m K(x, a), \end{aligned}$$

we have

$$y = f(x, a) = (x - x_0)^m \int_{a_0}^a K(x, a) da + L(x);$$

then

$$y_0 = f(x_0, a) = L(x_0),$$

and all the curves,  $y = f(x, a)$ , are seen to pass through the fixed point  $(x_0, y_0)$ . Hence the factor  $(x - x_0)^m$  gives rise to the so-called *point envelope*.

Equation  $c)$  presents two cases according as

$$F(x_0, a_0) \neq 0, \quad \text{or} \quad F(x_0, a_0) = 0.$$

If  $F(x_0, a_0) \neq 0$ , then  $F(x, a)$  is the function  $H(x, a)$  obtained by applying Weierstrass's Theorem to the function

$$(x - x_0)^m F(x, a).$$

Hence, if  $F(x_0, a_0) \neq 0$ , the factor  $F(x, a)$  gives rise to no envelope.

If  $F(x_0, a_0) = 0$ , Weierstrass's Theorem enables us to solve the equation

$$c) \quad F(x, a) = 0$$

for  $x$  as a function of  $a$ ,  $x = \phi(a)$ , in the neighborhood of  $a = a_0$ . The functions

$$x = \phi(a), \quad y = f[\phi(a), a] = \psi(a),$$

define an envelope of the curves

$$y = f(x, a)$$

in the region in question. These statements we now proceed to prove.

By Weierstrass's Theorem, we may write

$$F(x, a) \equiv \Omega(x, a) H(x, a),$$

where  $H(x, a)$  is analytic in the neighborhood of  $(x_0, a_0)$  and  $H(x_0, a_0) \neq 0$ . The  $x$ -roots of  $F(x, a) = 0$  in the neighborhood of  $a = a_0$  are all given as the roots of the equation

$$\Omega(x, a) = 0.$$

Let  $\Omega_i(x, a)$  be a factor of  $\Omega(x, a)$ , irreducible in the neighborhood of  $a_0$ ; it may be written

$$\Omega_i(x, a) \equiv (x - x_0)^l + A_1(a)(x - x_0)^{l-1} + \dots + A_l(a),$$

where  $A_j(a)$  is analytic at  $a_0$ ,  $A_j(a_0) = 0$ , and  $A_l(a) \neq 0$ . Then

$$\Omega_i(x, a) = 0$$

defines  $x$  as an  $l$ -valued function of  $a$ ,

$$x = \phi(a),$$

whose branches are all distinct except at  $a_0$ , where they are all connected in a single cycle. Cut each leaf of the Riemann surface along a ray emanating

from  $a_0$ , and denote the branches of  $\phi(a)$  corresponding to the severed leaves by  $\phi_1(a)$ ,  $\phi_2(a)$ ,  $\phi_3(a)$ ,  $\dots$ ,  $\phi_i(a)$ . The substitution of these functions in the equation

$$y = f(x, a)$$

gives

$$y = f[\phi_i(a), a] = \psi_i(a);$$

thus we obtain pairs of functions

$$x = \phi_i(a), \quad y = \psi_i(a),$$

such that in the region in question no two branches  $\phi_i(a)$ ,  $\phi_j(a)$  are the same.

In the neighborhood of any point  $a = a_1 \neq a_0$  in a suitably restricted vicinity of  $a_0$ , the functions,

$$x = \phi_i(a), \quad y = \psi_i(a),$$

define an envelope of the family  $y = f(x, a)$ .

For, from the character of  $F(x, a)$ , the function  $\phi_i(a)$  is non-constant, and  $\phi'_i(a) \neq 0$ ; hence the condition

$$|\phi'_i(a)| + |\psi'_i(a)| > 0$$

of the definition (§2) is fulfilled at any point of the region in question. The proof that the slopes of the curves

$$x = \phi_i(a), \quad y = \psi_i(a),$$

and

$$y = f(x, a)$$

are identical at an arbitrary point of the neighborhood of  $a_1$  is exactly that used in establishing the theorem of §3. Thus the functions  $x = \phi_i(a)$ ,  $y = \psi_i(a)$  meet all the requirements of the definition of §2 and the statement is established.

To the irreducible factor  $\Omega_i(x, a)$  there corresponds therefore an envelope consisting of a single branch analytic at  $a_0$ , or of several branches distinct and analytic in the neighborhood of  $a_0$  except at the point itself and meeting at the point. To the function  $\Omega(x, a)$  will correspond, of course, one envelope of this type for each of its distinct irreducible factors. These envelopes will, in general, be distinct; see §7, example 4, for an exception.

The results of the foregoing discussion are brought together in the follow-

ing fundamental theorem, which summarizes the facts concerning envelopes of the family of curves given in the explicit form  $y = f(x, a)$ .

**FUNDAMENTAL THEOREM.** *Given a one-parameter family of curves  $y = f(x, a)$ , where  $f(x, a)$  is an analytic function of  $x$  and  $a$  in the neighborhood of  $(x_0, a_0)$ , such that*

$$\begin{array}{ll} 1) & f_{a,x^i}(x_0, a) \equiv 0, & i = 0, 1, 2, \dots, m-1, \\ \text{but} & f_{a,x^m}(x_0, a) \neq 0, \end{array}$$

$$\begin{array}{ll} 2) & f_{a^j}(x, a_0) \equiv 0, & j = 1, 2, \dots, n, \\ \text{but} & f_{a^{n+1}}(x, a_0) \neq 0; \\ \text{then, if} & \end{array}$$

$$3) \quad f_{a^{n+1}, x^m}(x_0, a_0) \neq 0,$$

the curves  $y = f(x, a)$  have no envelope in the neighborhood of  $(x_0, y_0)$ , except a point envelope which occurs whenever  $m \geq 1$ ; but, if

$$3') \quad f_{a^{n+1}, x^m}(x_0, a_0) = 0,$$

the family  $y = f(x, a)$  has an envelope composed of one or more curves through the point  $(x_0, y_0)$ ; also, whenever  $m \geq 1$ , a point envelope at that point.

When  $m = 0$ , the notation  $i = 0, 1, 2, \dots, m-1$  is meaningless and 1) must be replaced by  $f_a(x_0, a) \neq 0$ .

**7. Examples.** *Example 1.* Examine for an envelope in the neighborhood of the point  $x_0 = 0, a_0 = 0$ , the curves:

$$y = f(x, a) \equiv 4a^5 - 5a^4x + 4a^6x^2 - 5a^4x^3 - x^9.$$

Here  $f(x, a)$  is analytic at  $(0, 0)$ , and

$$f_a(x, a) \equiv 20a^3(a - x)(1 + x^2),$$

so that  $f_a(x_0, a_0) = 0$  while  $f_{a^2}(x_0, a) \neq 0$ . The conditions of the Fundamental Theorem are fulfilled,  $n = 3, m = 0$ , and

$$\Omega(x, a) \equiv (a - x), \quad H(x, a) \equiv 20(1 + x^2).$$

Thus we obtain the envelope

$$x = a, \quad y = -a^5(1 + a + a^2)(1 - a + a^2),$$

consisting of one analytic branch through  $(0,0)$ ; the equation in Cartesian form is

$$x^9 + x^7 + x^5 + y = 0.$$

*Example 2.* Examine for an envelope in the neighborhood of  $x_0 = 0$ ,  $a_0 = 0$ , the curves:

$$y = f(x, a) \equiv 20x^4a^3 + x^7 + 15a^4x^2 + 15a^4x^4 + 12x^2a^5.$$

Here  $f_a(x, a) \equiv 60a^2x^2(x^2 + a)(1 + a),$

and we have the conditions of the Fundamental Theorem with  $m = 2$  and  $n = 2$ . Thus we obtain a point envelope at the origin, and an envelope which consists of a curve through  $(0,0)$  given by

$$x = \pm \sqrt{-a}, \quad y = 5a^5 + 3a^6 \mp a^3 \sqrt{-a},$$

or in Cartesian form:

$$y = x^7(1 - 5x^3 + 3x^5).$$

*Example 3.* Examine for an envelope in the neighborhood of  $x_0 = 0$ ,  $a_0 = 0$ , the curves:

$$y = f(x, a) \equiv x^2 \left[ ax^4 + 2a^2x^3 + \left( \frac{4}{3}a - \frac{1}{2} \right) a^2x^2 - \frac{4}{3}a^3x - a^4 \right].$$

Here  $f_a(x, a) \equiv x^2 \left[ x^4 + 4ax^3 + (4a^2 - a)x^2 - 4a^2x - 4a^3 \right]$   
 $\equiv x^2(x^2 - a)(x + 2a)^2.$

The envelope consists of the curves

$$x = a^{\frac{1}{3}}, \quad y = \frac{a^4}{6} \left[ 3 + 4a^{\frac{1}{3}} + 2a \right],$$

and

$$x = -2a, \quad y = \frac{4}{3}a^6 \left[ 112a - 17 \right];$$

the origin is a point envelope.

*Example 4.* Examine for an envelope in the neighborhood of  $x_0 = 0$ ,  $a_0 = 0$ , the curves:

$$y = f(x, a) \equiv (x - a)^2(x - 2a)^2.$$

Here  $f_a(x, a) \equiv 2(x - a)(x - 2a)(4a - 3x),$

and 
$$\Omega(x, a) \equiv (x - a)(x - 2a)\left(x - \frac{4}{3}a\right).$$

The Riemann surface consists of three distinct pieces, each composed of a single sheet. The envelope is made up of the loci

$$x = a, \quad y = 0; \quad x = 2a, \quad y = 0;$$

and 
$$x = \frac{4}{3}a, \quad y = \frac{4}{81}a^4.$$

This example illustrates how two different irreducible factors of  $\Omega(x, a)$  may give rise to the same envelope in the neighborhood of a point  $(x_0, y_0)$ .

## PART II. THE IMPLICIT CASE.

**8. Statement of the Problem.** Attention will now be directed to the case in which the family of plane curves is defined by an equation in the implicit form :

$$F(x, y, a) = 0,$$

where  $F(x, y, a)$  is an analytic function of  $x, y, a$  and where  $F_a(x, y, a) \neq 0$ . The definition of the envelope is the same as in §1. Two cases will be considered here: *a*) that in which the curves of the family have no multiple points in the region in question, *b*) that in which these curves have multiple points of order two only.

**9. A Sufficient Condition for an Envelope.** In Case *a*) the following theorem gives sufficient conditions for the existence of an envelope.

**THEOREM.** *If  $F(x, y, a)$  is a function, analytic in the three independent variables  $x, y$ , and  $a$ , at the point  $(x_0, y_0, a_0)$ , and if*

$$1) \quad F(x_0, y_0, a_0) = 0 \text{ and } F_a(x_0, y_0, a_0) = 0,$$

$$2) \quad \frac{D(F, F_a)}{D(x, y)} \equiv \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F_a}{\partial x} & \frac{\partial F_a}{\partial y} \end{vmatrix} \neq 0 \text{ at } (x_0, y_0, a_0),$$

$$3) \quad F_{aa}(x_0, y_0, a_0) \neq 0;$$

then the curves  $F(x, y, a) = 0$  have an envelope  $x = \phi(a)$ ,  $y = \psi(a)$ , which consists, in the neighborhood of the point  $(x_0, y_0, a_0)$ , of a single analytic curve.

*Proof:* The two functions  $F(x, y, a)$  and  $F_a(x, y, a)$  satisfy the conditions for obtaining  $x$  and  $y$  from the two simultaneous equations

$$F(x, y, a) = 0, \quad F_a(x, y, a) = 0$$

as given in the ordinary theorem on Implicit Functions.\* The solution of the equations gives rise to two analytic functions,

$$x = \phi(a), \quad y = \psi(a),$$

such that  $F[\phi(a), \psi(a), a] \equiv 0$  and  $F_a[\phi(a), \psi(a), a] \equiv 0$ , and that  $\phi(a_0) = x_0$  and  $\psi(a_0) = y_0$ . These two functions  $x = \phi(a)$ ,  $y = \psi(a)$  exhaust all the systems of values  $(x, y, a)$  in the neighborhood of  $(x_0, y_0, a_0)$  for which the functions  $F(x, y, a)$  and  $F_a(x, y, a)$  vanish simultaneously. Hence but a single curve is determined. Furthermore,  $\phi(a)$  and  $\psi(a)$  are not both constant; for differentiate the identity  $F_a(\phi, \psi, a) \equiv 0$ , thus obtaining

$$F_{ax}\phi'(a) + F_{ay}\psi'(a) + F_{aa} \equiv 0.$$

Then, if  $\phi(a)$  and  $\psi(a)$  were constant,  $\phi'(a) \equiv \psi'(a) \equiv 0$ , and  $F_{aa}[\phi(a), \psi(a), a] \equiv 0$  and, in particular,  $F_{aa}[\phi(a_0), \psi(a_0), a_0] = 0$ , in contradiction of condition 3). Therefore  $|\phi'(a)| + |\psi'(a)| > 0$ .

It remains merely to show that, at the common point  $(x_0, y_0, a_0)$ , the curves  $F(x, y, a_0) = 0$  and  $x = \phi(a)$ ,  $y = \psi(a)$  have the same slope. Differentiating the identity  $F(\phi, \psi, a) \equiv 0$ , we get

$$F_x\phi'(a) + F_y\psi'(a) + F_a \equiv 0.$$

But at the points under consideration  $F_a \equiv 0$ , and thus the last identity reduces to the form

$$F_x\phi'(a) + F_y\psi'(a) \equiv 0.$$

Now the slope of  $F(x, y, a_0) = 0$  is given by the equation  $F_x dx + F_y dy = 0$ , and  $F_x, F_y$  are not both zero, by 2), while the slope of  $x = \phi(a)$ ,  $y = \psi(a)$  is given by the equation  $\psi'(a) dx - \phi'(a) dy = 0$ , and the above identity is

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\* See Osgood: *Lehrbuch der Funktionentheorie*, vol. 1, pp. 52-55 and p. 345. The proof for a single equation is readily extended to a simultaneous system.

precisely the necessary and sufficient condition that these two equations, homogeneous in  $dx$  and  $dy$ , have a common solution other than  $(0, 0)$ . The theorem therefore is proved. The conditions are *sufficient*, but not *necessary*.

**10. Conditions for Double Points.** In case *b*) there will occur, in the pairs of solutions for  $x$  and  $y$  in terms of  $a$  from  $F(x, y, a) = 0$  and  $F_a(x, y, a) = 0$ , not only the envelope, in case there is one, but also the locus of double points. In the complex domain, we distinguish two classes of multiple points of order two, according as the tangents to the curve at the point are distinct or coincident. The first class we call *nodes*; the second class includes *cusps*, *tacnodes*, *points of osculation* and so forth. In the domain of reals, double points with distinct tangents are termed *nodes* or *conjugate points*, according as these tangents are real or imaginary. That we may get simple, definite conditions, we shall make in all the following discussion the restriction on the function  $F(x, y, a)$  that it involve no repeated factor. Then we may state the following theorem for a node.

**THEOREM I.** *Necessary and sufficient conditions that the locus of  $F(x, y) = 0$ , where  $F(x, y)$  is an analytic function of  $x$  and  $y$  in the neighborhood of  $(x_0, y_0)$ , consist, in this neighborhood, of two analytic branches through  $(x_0, y_0)$  with distinct slopes at this point, are that, at  $(x_0, y_0)$ ,*

$$1) F = 0, \quad 2) F_x = 0, \quad 3) F_y = 0, \quad 4) F_{xx}F_{yy} - F_{xy}^2 \neq 0.$$

*Proof.* There is no loss of generality in taking  $(x_0, y_0)$  as  $(0, 0)$ .

1. *The conditions are necessary.* Since  $(0, 0)$  is a point on the locus,  $F(0, 0) = 0$ . If either 2) or 3) is violated, the ordinary Implicit Function Theorem shows that the locus is a single analytic curve, contrary to the hypothesis. The function  $F(x, y)$  may, therefore, be written in the form:

$$\begin{aligned} F(x, y) &\equiv ax^2 + 2bxy + cy^2 + dx^3 + 3ex^2y + \dots \\ &\equiv x^2 \left( a + 2b \frac{y}{x} + c \frac{y^2}{x^2} \right) + x^3 \left( d + 3e \frac{y}{x} + \dots \right) + \dots \\ &\equiv x^2 f(x, \lambda), \quad \lambda = \frac{y}{x}. \end{aligned}$$

The only solution of  $F(0, y) = 0$  is  $y = 0$ . Consider a particular branch of the locus through  $(0, 0)$ ; its slope is

$$\lim_{x \rightarrow 0} \frac{y}{x} = \lim_{x \rightarrow 0} \lambda,$$

$y$  and  $\lambda$  being considered as functions of the independent variable  $x$ . The variables  $x$  and  $\lambda$  are connected by the equation

$$f(x, \lambda) \equiv a + 2b\lambda + c\lambda^2 + x(d + 3e\lambda + \dots) + \dots = 0.$$

Moreover  $\lim x = 0$ ; consequently the limiting value of  $\lambda$  satisfies the equation

$$f(0, \lambda) \equiv a + 2b\lambda + c\lambda^2 = 0.$$

Since, by hypothesis, the slopes,  $\lambda_1, \lambda_2$ , at  $(0, 0)$  are distinct,  $b^2 - ac \neq 0$ . But  $F_{xx}(0, 0) = 2a$ ,  $F_{xy}(0, 0) = 2b$ ,  $F_{yy}(0, 0) = 2c$ ; hence the quadratic equation in  $\lambda$  is equivalent to either of the two equations,

$$ax^2 + 2bxy + cy^2 = 0, \text{ and } F_{xx}(dx)^2 + 2F_{xy}dx dy + F_{yy}(dy)^2 = 0,$$

and the condition just obtained becomes

$$4) \quad F_{xx} F_{yy} - F_{xy}^2 \neq 0.$$

This establishes the necessary conditions.

2. *The conditions are sufficient.* Because of 1), 2), 3), we have, as above,

$$F(x, y) \equiv ax^2 + 2bxy + cy^2 + dx^3 + 3ex^2y + \dots$$

$$\equiv x^2 f(x, \lambda), \quad \lambda = \frac{y}{x},$$

where, because of 4),

$$b^2 - ac \neq 0.$$

The only solution of  $F(0, y) = 0$  is  $y = 0$ . Consider, then, the function

$$f(x, \lambda) \equiv a + 2b\lambda + c\lambda^2 + x(d + 3e\lambda + \dots) + \dots$$

$$\equiv A_0(\lambda) + A_1(\lambda) \cdot x + A_2(\lambda) \cdot x^2 + \dots$$

The roots,  $\lambda_1, \lambda_2$ , of  $A_0(\lambda) = 0$  are distinct; hence

$$\frac{\partial f(0, \lambda_i)}{\partial \lambda} = 2(b + c\lambda_i) \neq 0, \quad i = 1, 2.$$

Therefore

$$f(x, \lambda) = 0$$

determines  $\lambda = \phi_i(x)$ ,  $i = 1, 2$ ,  $[\phi_i(0) = \lambda_i, f(x, \phi_i(x) \equiv 0]$ , a pair of functions, each single valued and continuous in the neighborhood of  $x = 0$ , and having there a continuous first derivative. Then  $y = \lambda x$  gives two functions

$$y = x\phi_1(x) = \phi(x), \quad y = x\phi_2(x) = \psi(x),$$

which define two analytic curves through  $(0, 0)$ .

The slopes of the curves at  $(0, 0)$  are

$$\left. \frac{dy}{dx} \right|_{x=x_0=0} = x_0\phi'_i(x_0) + \phi_i(x_0) = \lambda_i, \quad i = 1, 2.$$

But  $\lambda_1 \neq \lambda_2$ ; hence the two curves through  $(0, 0)$  have there distinct slopes.

The curves  $\lambda = \phi_i(x)$  exhaust all points in the neighborhood of  $(0, \lambda_i)$  for which  $f(x, \lambda)$  vanishes. To show that these curves exhaust all points in the neighborhood of  $x = 0, \lambda = \lambda$ , for which  $f(x, \lambda)$  vanishes, proceed as follows. Take  $\lambda = \lambda_3$  any point not in the neighborhood of either  $\lambda_1$  or  $\lambda_2$ . Write  $f(x, \lambda)$  as

$$f(x, \lambda) \equiv A_0(\lambda) + \sum_{m=1}^{\infty} A_m(\lambda)x^m.$$

Then  $|A_0(\lambda_3)| > G$ ,  $G$  being some fixed positive number; moreover, there exists a positive constant  $\rho$ , such that

$$\left| \sum_{m=1}^{\infty} A_m(\lambda_3) x^m \right| < G, \quad \text{for } |x| < \rho.$$

Hence this value pair  $(x, \lambda_3)$  does not cause  $f(x, \lambda)$  to vanish. This shows that the complete locus of  $F(x, y) = 0$  is represented, in the neighborhood of  $(0, 0)$ , by the two curves

$$y = \phi(x), \quad y = \psi(x),$$

and the theorem is proved.

**COROLLARY.** *If  $F(x, y)$  is real when  $x$  and  $y$  are real, necessary and sufficient conditions that  $F(x, y) = 0$  define at  $(x_0, y_0)$  two analytic curves with distinct slopes which are a) real, b) imaginary, are conditions 1), 2), 3) above, with 4) replaced by*

$$4a) \quad F_{xx}F_{yy} - F_{xy}^2 < 0, \quad 4b) \quad F_{xx}F_{yy} - F_{xy}^2 > 0,$$

respectively.

**THEOREM II.** *Necessary and sufficient conditions that the locus of  $F(x, y) = 0$ , where  $F(x, y)$  is an analytic function of  $x$  and  $y$  in the neighborhood of  $(x_0, y_0)$ , consist, in this neighborhood, of two branches through  $(x_0, y_0)$  with coincident slopes at this point, are that, at  $(x_0, y_0)$ ,*

$$1) F = 0, \quad 2) F_x = 0, \quad 3) F_y = 0, \quad 4) F_{xx}F_{yy} - F_{xy}^2 = 0,$$

while 5)  $F_{xx}, F_{xy}, F_{yy}$

do not all vanish. These branches may be connected or not.

*Proof.* As before let us take  $(x_0, y_0)$  as  $(0, 0)$ .

*Necessary Conditions.* These are established precisely as in the foregoing theorem.

*Sufficient Conditions.* Because of 1), 2), 3), we have

$$F(x, y) \equiv ax^2 + 2bxy + cy^2 + dx^3 + 3ex^2y + \dots,$$

where, because of 4), 5),

$$b^2 - ac = 0, \quad a, b, c, \text{ not all zero.}$$

Since either  $a \neq 0$  or  $c \neq 0$ , let us assume that  $c \neq 0$ . Then

$$F(0, y) \neq 0, \quad F(0, 0) = 0, \quad F_{y,y}(0, 0) = 2c \neq 0.$$

Hence we have, by Weierstrass's Theorem,

$$F(x, y) \equiv \Omega(x, y) H(x, y),$$

where  $H(x, y)$  is analytic at  $(0, 0)$ ,  $H(0, 0) \neq 0$ , and where

$$\Omega(x, y) \equiv y^2 + A_1(x)y + A_2(x),$$

$$A_i(x) \text{ analytic at } x = 0, \quad A_i(0) = 0.$$

The locus of  $\Omega(x, y) = 0$  exhausts all points in the neighborhood of  $(0, 0)$  for which  $F(x, y)$  vanishes. We may write

$$\begin{aligned} \Omega(x, y) &\equiv y^2 + A_1(x)y + A_2(x) \\ &\equiv \left(y + \frac{A_1 - B}{2}\right) \left(y + \frac{A_1 + B}{2}\right), \quad B^2 \equiv A_1^2 - 4A_2, \end{aligned}$$

where  $B^2(x) \neq 0$ , because  $F(x, y)$  has, by hypothesis, no repeated factor. If

the expansion of  $B^2(x)$  about  $x = 0$  begins with an even power of  $x$ ,  $B(x)$  is analytic at  $x = 0$ ,  $\Omega(x, y)$  is reducible, and  $\Omega(x, y) = 0$  defines two branches, analytic at  $(0, 0)$ . This is the case at a tacnode. If the expansion of  $B^2(x)$  about  $x = 0$  begins with an odd power of  $x$ ,  $B(x)$  is not analytic at the point,  $\Omega(x, y)$  is irreducible, and  $\Omega(x, y) = 0$  defines two branches, analytic in the neighborhood of  $(0, 0)$ , but connected at that point.

If  $c = 0$ , then  $a \neq 0$ , Weierstrass's Theorem gives

$$\Omega(x, y) \equiv x^2 + B_1(y)x + B_2(y),$$

and  $\Omega(x, y) = 0$  defines two branches as before.

The slopes, at  $(0, 0)$ , of the locus of  $F(x, y) = 0$  are determined, as in Theorem I, by the equation

$$a + 2b\lambda + c\lambda^2 = 0.$$

Since, by hypothesis,  $b^2 - ac = 0$ , these two slopes are equal, and the theorem is proved.

**11. The Nodal Locus.** THEOREM I. *A necessary condition that a given locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a nodal locus of the curves of the family  $F(x, y, a) = 0$  is that the relations*

$$\begin{aligned} 1) \quad & F \equiv 0, \quad 2) \quad F_x \equiv 0, \quad 3) \quad F_y \equiv 0, \quad 4) \quad F_a \equiv 0, \\ 5) \quad & F_{ax} \neq 0 \quad \text{or} \quad F_{ay} \neq 0, \quad 6) \quad F_{xx}F_{yy} - F_{xy}^2 \neq 0, \end{aligned}$$

$$7) \quad \begin{vmatrix} F_{xx} & F_{xy} & F_{xa} \\ F_{yx} & F_{yy} & F_{ya} \\ F_{ax} & F_{ay} & F_{aa} \end{vmatrix} \equiv 0,$$

be satisfied along this locus.

*Proof.* Since  $x_i = \mu(a_i)$ ,  $y_i = \nu(a_i)$  is a node of the curve  $a_i$ , therefore, by the preceding section

$$F = 0, \quad F_x = 0, \quad F_y = 0, \quad F_{xx}F_{yy} - F_{xy}^2 \neq 0,$$

at this point. Since these conditions are true at every point of the locus, then

$$1) \quad F \equiv 0, \quad 2) \quad F_x \equiv 0, \quad 3) \quad F_y \equiv 0, \quad 6) \quad F_{xx}F_{yy} - F_{xy}^2 \neq 0,$$

along the locus  $x = \mu(a)$ ,  $y = \nu(a)$ . From the first identity,

$$F_x dx + F_y dy + F_a da \equiv 0$$

along the same locus. But since  $F_x \equiv 0$  and  $F_y \equiv 0$  here, it follows that

4)  $F_a(x, y, a) \equiv 0$  along this locus. Differentiating the identities 2), 3), 4), we obtain

$$F_{xx}\mu' + F_{xy}\nu' + F_{xa} \equiv 0,$$

$$F_{yx}\mu' + F_{yy}\nu' + F_{ya} \equiv 0,$$

$$F_{ax}\mu' + F_{ay}\nu' + F_{aa} \equiv 0,$$

along  $x = \mu(a)$ ,  $y = \nu(a)$ . A necessary condition that these be consistent is

$$7) \quad \begin{vmatrix} F_{xx} & F_{xy} & F_{xa} \\ F_{yx} & F_{yy} & F_{ya} \\ F_{ax} & F_{ay} & F_{aa} \end{vmatrix} \equiv 0.$$

Because of 6), the equations

$$F_{xx}\mu' + F_{xy}\nu' + F_{xa} = 0,$$

$$F_{yx}\mu' + F_{yy}\nu' + F_{ya} = 0,$$

determine  $\mu'$  and  $\nu'$  at any point along  $x = \mu(a)$ ,  $y = \nu(a)$ , as

$$(A) \quad \mu' = \frac{F_{xy} F_{ya} - F_{yy} F_{xa}}{F_{xx} F_{yy} - F_{xy}^2}, \quad \nu' = \frac{F_{xy} F_{xa} - F_{xx} F_{ya}}{F_{xx} F_{yy} - F_{xy}^2}.$$

Since  $\mu'(a)$  and  $\nu'(a)$  are not both identically zero it follows that at least one of the inequalities  $F_{ax}(x, y, a) \neq 0$ ,  $F_{ay}(x, y, a) \neq 0$  must hold along  $x = \mu(a)$ ,  $y = \nu(a)$ . This establishes the theorem.

**THEOREM II.** *A sufficient condition that the given locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a nodal locus of the curves  $F(x, y, a) = 0$  is that the relations*

$$1) F \equiv 0, \quad 2) F_x \equiv 0, \quad 3) F_y \equiv 0, \quad 4) F_{xx}F_{yy} - F_{xy}^2 \neq 0,$$

*be satisfied along this locus.*

*Proof.* The conditions of the theorem of §10 are fulfilled at every point of the given locus. Hence the theorem follows at once.

THEOREM III. *If, at  $(x_0, y_0, a_0)$ ,*

$$1) F_x = 0, \quad 2) F_y = 0, \quad 3) F_{xx}F_{yy} - F_{xy}^2 \neq 0,$$

*then*  $F_x(x, y, a) = 0, \quad F_y(x, y, a) = 0$

*define*  $x = \mu(a), \quad y = \nu(a),$

*where*  $a) \quad \mu(a), \quad \nu(a) \text{ are analytic at } a_0,$

$$b) \quad \mu(a_0) = x_0, \quad \nu(a_0) = y_0.$$

*If, furthermore,*

$$4) F(\mu, \nu, a) \equiv 0, \quad 5) F_{ax}(\mu, \nu, a) \neq 0, \text{ or } F_{ay}(\mu, \nu, a) \neq 0,$$

*then*  $c) \quad |\mu'(a)| + |\nu'(a)| \neq 0,$

*and*  $x = \mu(a), \quad y = \nu(a)$

*define, in the neighborhood of  $(x_0, y_0, a_0)$ , a nodal locus of the curves  $F(x, y, a) = 0$ .*

*Proof.* Conditions 1), 2), 3) insure, by the ordinary Implicit Function Theorem, the existence of two functions  $x = \mu(a)$ ,  $y = \nu(a)$  satisfying conclusions a) and b), and such that

$$1') F_x(\mu, \nu, a) \equiv 0, \quad 2') F_y(\mu, \nu, a) \equiv 0.$$

That conditions 3), 5) are sufficient that  $\mu'(a)$ ,  $\nu'(a)$  do not both vanish identically, follows at once from the equations (A) for  $\mu'(a)$ ,  $\nu'(a)$  obtained in the proof of Theorem I, above. Finally, 4), 1'), 2'), 3) constitute precisely the sufficient conditions of Theorem II above, which completes the proof of the Theorem.

THEOREM IV.\* *A necessary and sufficient condition that a given nodal locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , of the curves  $F(x, y, a) = 0$  be an envelope of one of the branches at the node, is that*

$$F_{aa}(\mu, \nu, a) \equiv 0.$$

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\* This theorem is due to M. J. M. Hill, *Proc. London Math. Soc.*, vol. 22 (1891), p. 222.

*Proof. Necessary condition.* Since  $x = \mu(a)$ ,  $y = \nu(a)$  is a nodal locus of  $F(x, y, a) = 0$ ,

then  $F_x(\mu, \nu, a) \equiv 0$ ,  $F_y(\mu, \nu, a) \equiv 0$ ,  $F_a(\mu, \nu, a) \equiv 0$ ;

$$1) \quad F_{xx}dx + F_{xy}dy + F_{xa}da \equiv 0,$$

$$2) \quad F_{yx}dx + F_{yy}dy + F_{ya}da \equiv 0,$$

$$3) \quad F_{ax}dx + F_{ay}dy + F_{aa}da \equiv 0,$$

are consistent along the given locus.

Since, by hypothesis, the slope of the nodal locus is equal to the slope of one of the branches at the node,  $dy/dx$  from the above set must satisfy the equation

$$4) \quad F_{xx}(dx)^2 + 2F_{xy}dx \cdot dy + F_{yy}(dy)^2 = 0,$$

which determines the slopes of the two branches at the node. Multiply 1) by  $dx$ , 2) by  $dy$ , and add:

$$5) \quad F_{xx}(dx)^2 + 2F_{xy}dx \cdot dy + F_{yy}(dy)^2 + F_{xa}dx \cdot da + F_{ya}dy \cdot da \equiv 0,$$

whence

$$da(F_{xa}dx + F_{ya}dy) \equiv 0$$

along the locus. Since  $da \neq 0$ , it follows that

$$6) \quad F_{xa}dx + F_{ya}dy \equiv 0$$

along the locus. Hence, from 3), it follows that  $F_{aa}(\mu, \nu, a) \equiv 0$ , as was to be proved.

*Sufficient condition.* Since  $x = \mu(a)$ ,  $y = \nu(a)$  is a nodal locus, 1), 2), 3), and therefore 5) are true. Since, by hypothesis,  $F_{aa}(\mu, \nu, a) \equiv 0$ , 3) insures that 6) is true; 5) then reduces to 4); that is,  $dy/dx$  from set 1), 2), 3) satisfies 4), and the locus is an envelope of a branch of  $F(x, y, a) = 0$  through the node. This completes the proof of the theorem.

**12. Conditions for the Real Case.** If  $F(x, y, a)$  is real when  $x, y, a$  are real, the theorems of §11 read for  $a$ ) nodes,  $b$ ) conjugate points, according as we change the condition  $F_{xx}F_{yy} - F_{xy}^2 \neq 0$  to read

$$a) \quad F_{xx}F_{yy} - F_{xy}^2 < 0, \quad b) \quad F_{xx}F_{yy} - F_{xy}^2 > 0, \text{ respectively.}$$

*Example 1.* Consider the family of lemniscates :

$$F(x, y, a) \equiv \{(x+a)^2 + (y-a)^2\}^2 - 4b^2\{(x+a)^2 - (y-a)^2\} = 0.$$

The nodal locus is the straight line  $x = -a, y = a$ .

$$F_{aa}(x, y, a) \equiv 8 \{(x+a)^2 + (y-a)^2 + (x-y+2a)^2\}.$$

Since  $F_{aa}(\mu, \nu, a) \equiv 0$ , the nodal locus in an envelope of one of the branches through the node.

*Example 2.* Consider the family of curves

$$F(x, y, a) \equiv x^3 + (y-a)^3 - 3bx(y-a) = 0.$$

The nodal locus is  $x = 0, y = a$ .

$$F_{aa}(x, y, a) \equiv 6(y-a), \quad F_{aa}(\mu, \nu, a) \equiv 0.$$

Hence the nodal locus is an envelope.

### 13. The Locus of Double Points with Coincident Tangents.

As stated in §10, Theorem II, necessary and sufficient conditions that the point  $(x_0, y_0)$  be, on the curve  $F(x, y) = 0$ , a double point with coincident tangents are that

$$F = 0, \quad F_x = 0, \quad F_y = 0, \quad F_{xx}F_{yy} - F_{xy}^2 = 0,$$

at the point, while  $F_{xx}, F_{xy}, F_{yy}$  do not all vanish there.

**THEOREM I.** *Necessary conditions that a given locus  $x = \mu(a), y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a locus of double points with coincident tangents for the curves  $F(x, y, a) = 0$  are :*

$$1) \ F \equiv 0, \quad 2) \ F_x \equiv 0, \quad 3) \ F_y \equiv 0, \quad 4) \ F_a \equiv 0,$$

$$5) \ F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya} \equiv F_{ax} : F_{ay} : F_{aa},$$

along the locus, while

$$6) \quad F_{xx}, \ F_{xy}, \ F_{yy}$$

do not all vanish simultaneously at any point of the given locus.

*Proof.* Conditions 1), 2), 3), 6), and

$$7) \quad F_{xx}F_{yy} - F_{xy}^2 \equiv 0$$

follow at once from the fact that the above requirements for a double point with coincident tangents hold at every point of the given locus. Differentiating 1), we obtain condition 4) by using 2) and 3). From the identities 2), 3), 4), we obtain the consistent equations

$$\begin{aligned} 8) \quad & F_{xx}\mu' + F_{xy}\nu' + F_{xa} = 0, \\ 9) \quad & F_{yx}\mu' + F_{yy}\nu' + F_{ya} = 0, \\ 10) \quad & F_{ax}\mu' + F_{ay}\nu' + F_{aa} = 0. \end{aligned}$$

From 7), and the consistency of 8) and 9), it follows that

$$11) \quad F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya}.$$

From  $F_{xx} : F_{xy} \equiv F_{xa} : F_{ya}$ , and the consistency of 8) and 10), it follows that

$$12) \quad F_{xx} : F_{xy} : F_{xa} \equiv F_{ax} : F_{ay} : F_{aa}.$$

Condition 5) is obtained by combining 11) and 12). This completes the proof of the theorem.

**THEOREM II.** *Sufficient conditions that a given locus  $x = \mu(a)$ ,  $y = \nu(a)$ , where  $|\mu'(a)| + |\nu'(a)| \neq 0$ , be a locus of double points with coincident tangents for the curves  $F(x, y, a) = 0$ , are:*

$$1) \ F \equiv 0, \quad 2) \ F_x \equiv 0, \quad 3) \ F_y \equiv 0, \quad 4) \ F_{xx} F_{yy} - F_{xy}^2 \equiv 0,$$

along the locus, while

$$5) \quad F_{xx}, F_{xy}, F_{yy}$$

do not all vanish simultaneously at any point of the given locus.

*Proof.* The conditions for such a point are fulfilled at every point of the given locus, and the theorem is established.

**THEOREM III.** *A necessary and sufficient condition that a given locus of double points with coincident tangents,*

$$\mu(a_0) = x_0, \quad \nu(a_0) = y_0, \quad |\mu'(a)| + |\nu'(a)| \neq 0,$$

of the curves  $F(x, y, a) = 0$ , be an envelope of these curves is that

$$F_{aa}(\mu, \nu, a) \equiv 0.$$

*Proof.* Since  $x = \mu(a)$ ,  $y = \nu(a)$  is a locus of double points with coincident tangents, we have along it

$$1) F \equiv 0, \quad 2) F_x \equiv 0, \quad 3) F_y \equiv 0, \quad 4) F_a \equiv 0,$$

$$5) F_{xx} : F_{xy} : F_{xa} \equiv F_{yx} : F_{yy} : F_{ya} \equiv F_{ax} : F_{ay} : F_{aa},$$

while

$$6) F_{xx}, F_{xy}, F_{yy}$$

do not all vanish at any point of the given locus.

*Necessary condition.* Since the slope of the given locus is, by hypothesis, equal at every point to the slope of the double point tangent at that point, we have, using 5) :

$$\nu'(a) : \mu'(a) \equiv -F_{xx}(\mu, \nu, a) : F_{xy}(\mu, \nu, a)$$

$$\equiv -F_{yx}(\mu, \nu, a) : F_{yy}(\mu, \nu, a)$$

$$\equiv -F_{ax}(\mu, \nu, a) : F_{ay}(\mu, \nu, a);$$

$$\text{hence } 7) F_{xx}\mu' + F_{xy}\nu' \equiv 0,$$

$$8) F_{yx}\mu' + F_{yy}\nu' \equiv 0,$$

$$9) F_{ax}\mu' + F_{ay}\nu' \equiv 0,$$

are necessary conditions along the locus.

From these and the identities 8), 9), 10) under Theorem I, it follows that

$$F_{ax}(\mu, \nu, a) \equiv 0, \quad F_{ay}(\mu, \nu, a) \equiv 0, \quad F_{aa}(\mu, \nu, a) \equiv 0.$$

This establishes the necessary condition, and incidentally shows that  $F_{ax}(\mu, \nu, a) \equiv 0$  and  $F_{ay}(\mu, \nu, a) \equiv 0$  are equivalent necessary conditions.

*Sufficient conditions.* Let  $F_{aa}(\mu, \nu, a) \equiv 0$ ; then, from 5)

$$F_{ax}(\mu, \nu, a) \equiv 0, \quad F_{ay}(\mu, \nu, a) \equiv 0.$$

The equations 7), 8), 9) are then true, and, because of 5),

$$\nu'(a) : \mu'(a) \equiv -F_{xx}(\mu, \nu, a) : F_{xy}(\mu, \nu, a)$$

$$\equiv -F_{yx}(\mu, \nu, a) : F_{yy}(\mu, \nu, a),$$

that is, the slope of the given locus is equal to that of the double point tangent, and the locus is an envelope. This completes the proof of the theorem.

**14. Examples.** *Example 1.* Consider the curves

$$F(x, y, a) \equiv (x - y)^2 - (x + y - 2a)^3 = 0.$$

The cusp locus is  $x = a, y = a$ .

$$F_{aa}(x, y, a) \equiv -24(x + y - 2a).$$

Since  $F_{aa}(a, a, a) \equiv 0$ , the cusp locus is an envelope.

*Example 2.* Consider the family of cissoids

$$F(x, y, a) \equiv x^3 + (x - 1)(y - a)^2 = 0.$$

The cusp locus is  $x = 0, y = a$ .

Since  $F_{aa}(x, y, a) \equiv 2(x - 1) \neq 0$  along the cusp locus, this locus is not an envelope.

*Example 3.* Consider the curves

$$F(x, y, a) \equiv y^2 - 2y(x - a)^2 + (x - a)^4 - (x - a)^5 = 0.$$

The cusp locus is  $x = a, y = 0$ . These cusps are of the second kind, both branches of the curve lying on the same side of the cusp tangent.

$$F_{aa}(x, y, a) \equiv -4y + 12(x - a)^2 - 20(x - a)^3.$$

Since  $F_{aa}(\mu, \nu, a) \equiv 0$ , the cusp locus is an envelope.

*Example 4.* Consider the curves

$$F(x, y, a) \equiv (x + y)^2 - (x - y - 2a)^4 + (x - y - 2a)^6 = 0.$$

The line  $x = a, y = -a$ , is the locus of tacnodes of the curves. Since  $F_{aa}(x, y, a) \equiv -48(x - y - 2a)^2 + 120(x - y - 2a)^4 \equiv 0$  along this locus of tacnodes, the locus is an envelope of the curves.

*Example 5.* Consider the curves

$$F(x, y, a) \equiv (y - a)^2 + (x - a)^4 + (y - a)^4 = 0.$$

Each curve  $a$  of this family has but one real point,  $x = a, y = a$ ;

$$F_{xx}(x, y, a) \equiv 12(x - a)^2, F_{xy}(x, y, a) \equiv 0, F_{yy}(x, y, a) \equiv 2 + 12(y - a)^2.$$

The conditions for double points are satisfied at every point of  $x = a$ ,  $y = a$ , which is therefore the locus of isolated double points of the given curves. The equation for the slopes of the branches through such a point,

$$(x - x_0)^2 F_{xx} + 2(x - x_0)(y - y_0) F_{xy} + (y - y_0)^2 F_{yy} = 0,$$

shows that the slopes of the two branches through the point  $(a_0, a_0)$  are both zero; we shall call this the slope of the coincident tangents to the curves at the isolated double points.

$$F_{aa}(x, y, a) \equiv 2 + 12(x - a)^2 + 12(y - a)^2.$$

Since  $F_{aa}(\mu, \nu, a) \equiv 2 \neq 0$ , the locus of isolated points is not an envelope.

The family

$$F(x, y, a) \equiv y^2 + (x - a)^4 + y^4 = 0$$

has  $x = a$ ,  $y = 0$  as a locus of isolated double points with coincident slopes. Since  $F_{aa}(a, 0, a) \equiv 0$ , this locus is an envelope.